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# Free nilpotent Lie algebras admitting ad-invariant metrics

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In this work we find necessary and sufficient conditions for a free nilpotent or a free metabelian nilpotent Lie algebra to be endowed with an ad-invariant metric. For such nilpotent Lie algebras admitting an ad-invariant metric the corresponding automorphisms groups are studied.



## 1. Introduction

An ad-invariant metric on a Lie algebra $\mathfrak{g}$ is a nondegenerate symmetric bilinear form $\langle\,,\rangle$ which satisfies

$$\langle [x,y],z\rangle + \langle y,[x,z]\rangle = 0 \quad \text{for all } x,y,z\in\mathfrak{g}. \tag{1}$$

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Lie algebras endowed with ad-invariant metrics (also called "metric" or "quadratic") became relevant some years ago when they were useful in the formulation of some physical problems such as the known Adler–Kostant–Symes scheme. They also constitute the basis for the construction of bialgebras and they give rise to interesting pseudo-Riemannian geometry [5]. For instance in [22] a result originally due to Kostant [16] was revalidated for pseudo-Riemannian metrics: it states that the Lie algebra of the isometry group of a naturally reductive pseudo-Riemannian space (in particular symmetric spaces) can be endowed with an ad-invariant metric.

Semisimple Lie algebras are examples of Lie algebras admitting an ad-invariant metric since the Killing form is nondegenerate. In the solvable case, the Killing form is degenerate so one must search for another bilinear form with the ad-invariance property. The first investigations concerning general Lie algebras with ad-invariant metrics appeared in [9,17]. They get structure results proposing a method to construct these Lie algebras recursively. This enables a classification of nilpotent Lie algebras admitting ad-invariant metrics of dimension $\leqslant 7$ in [9] and a determination of the Lorentzian Lie algebras in [18]. The point is that by this recursive method one can reach the same Lie algebra starting from two non-isomorphic Lie algebras. This fact makes difficult the classification in higher dimensions. More recently a new proposal for the classification problem is presented in [13] and this is applied in [14] to get the nilpotent Lie algebras with ad-invariant metrics of dimension $\leqslant 10$.

However the basic question whether a non-semisimple Lie algebra admits such a metric is still opened. In the present paper we deal with this problem in the family of free nilpotent and free metabelian nilpotent Lie algebras.

**Theorem 3.8.** *Let $\mathfrak{n}_{m,k}$ be the free $k$-step nilpotent Lie algebra on $m$ generators. Then $\mathfrak{n}_{m,k}$ admits an ad-invariant metric if and only if $(m,k)=(3,2)$ or $(m,k)=(2,3)$.*

The techniques for the proof do not make use of the extension procedures mentioned before, but properties of free nilpotent Lie algebras which combined with the ad-invariance condition enable the deduction of the Lie algebras $\mathfrak{n}_{2,3}$ and $\mathfrak{n}_{3,2}$. We note that for $k=2,3$ the free and free metabelian $k$-step nilpotent Lie algebras coincide. For the free metabelian case the first approach lies in the fact that 2-step solvable Lie algebras admitting ad-invariant metrics are nilpotent and at most 3-step (Lemma 3.3). Thus working out we get the next result.

**Theorem 3.9.** *Let $\tilde{\mathfrak{n}}_{m,k}$ be the free metabelian $k$-step nilpotent Lie algebra on $m$ generators. Then $\tilde{\mathfrak{n}}_{m,k}$ admits an ad-invariant metric if and only if $(m,k)=(3,2)$ or $(m,k)=(2,3)$.*

These two Lie algebras have been studied since a long time in sub-Riemannian geometry [20]. Thus $\mathfrak{n}_{2,3}$ is associated to the Carnot group distribution (see for instance [2]), which is related to the "rolling balls problem", treated by Cartan in [4]. The prolongation, representing – roughly speaking – the maximal possible symmetry of the distribution, in the case of $\mathfrak{n}_{2,3}$ is the exceptional Lie algebra $\mathfrak{g}_2$ [2]. The Lie algebra $\mathfrak{n}_{3,2}$ was studied more recently in [21] in the context of the geometric characterization of the so-called Maxwell set, wave fronts and caustics (see also [19]).

We complete the work with a study of the group of automorphisms of the Lie algebras $\mathfrak{n}_{2,3}$ and $\mathfrak{n}_{3,2}$. The corresponding structure is described and in particular the subgroup of orthogonal automorphisms is determined.

Following [9] and the considerations above all Lie algebras here are over a field $K$ of characteristic 0.

## 2. Free and free metabelian nilpotent Lie algebras

Let $\mathfrak{g}$ denote a Lie algebra. The so-called central descending and ascending series of $\mathfrak{g}$, respectively $\{C^r(\mathfrak{g})\}$ and $\{C_r(\mathfrak{g})\}$ for all $r \geqslant 0$, are constituted by the ideals in $\mathfrak{g}$, which for non-negative integers $r$, are given by

$$\begin{aligned} C^0(\mathfrak{g}) &= \mathfrak{g}, & C_0(\mathfrak{g}) &= 0, \\ C^r(\mathfrak{g}) &= \left[\mathfrak{g}, C^{r-1}(\mathfrak{g})\right], & C_r(\mathfrak{g}) &= \left\{x \in \mathfrak{g}\colon [x,\mathfrak{g}] \in C_{r-1}(\mathfrak{g})\right\}. \end{aligned}$$

Note that $C_1(\mathfrak{g})$ is by definition the center of $\mathfrak{g}$, which will be denoted by $\mathfrak{z}(\mathfrak{g})$.

A Lie algebra $\mathfrak{g}$ is called *k-step nilpotent* if $C^k(\mathfrak{g}) = \{0\}$ but $C^{k-1}(\mathfrak{g}) \neq \{0\}$ and clearly $C^{k-1}(\mathfrak{g}) \subseteq \mathfrak{z}(\mathfrak{g})$.

**Example 2.1.** Heisenberg Lie algebras. Let $X_1, \dots, X_n, Y_1, \dots, Y_n$ denote a basis of the $2n$-dimensional real vector space $V$ and let $Z \notin V$. Define $[X_i, Y_j] = \delta_{ij} Z$ and $[Z, U] = 0$ for all $U \in V$. Thus $\mathfrak{h}_n = V \oplus \mathbb{R}Z$ is the (real) Heisenberg Lie algebra of dimension $2n+1$, which is 2-step nilpotent.

We shall make use of the notation $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}'' = [\mathfrak{g}', \mathfrak{g}']$. A Lie algebra is called *2-step solvable* if its commutator is abelian, that is $\mathfrak{g}'' = 0$.

Let $\mathfrak{f}_m$ denote the free Lie algebra on $m$ generators, with $m \geqslant 2$. (Notice that a unique element spans an abelian Lie algebra.) Thus

- the *free metabelian k-step nilpotent* Lie algebra on $m$ generators is defined as

$$\tilde{\mathfrak{n}}_{m,k} := \mathfrak{f}_m / \big(C^k(\mathfrak{f}_m) + \mathfrak{f}_m''\big),$$

- the *free k-step nilpotent* Lie algebra on $m$ generators $\mathfrak{n}_{m,k}$ is defined as the quotient algebra

$$\mathfrak{n}_{m,k} := \mathfrak{f}_m / C^k(\mathfrak{f}_m).$$

In particular free metabelian nilpotent Lie algebras of any degree are 2-step solvable.

**Remark 1.** For $k = 2, 3$ any $k$-step nilpotent Lie algebra is 2-step solvable, which follows from the Jacobi identity. Thus for the free nilpotent ones we get $\tilde{\mathfrak{n}}_{m,k} = \mathfrak{n}_{m,k}$ for $k = 2, 3$.

Let $\mathfrak{n}_{m,k}$ be a free $k$-step nilpotent Lie algebra and let $\{e_1, \dots, e_m\}$ be an ordered set of generators. In [11] Hall gives a basis of $\mathfrak{n}_{m,k}$ whose elements are monomials in the generators. The *length* of an element in a Hall basis is the degree of the monomial and will be denoted by $\ell$ (see [6,10] for details).

Each Hall basis gives rise to a natural graduation of $\mathfrak{n}_{m,k}$:

$$\mathfrak{n}_{m,k} = \bigoplus_{s=1}^{k} \mathfrak{p}(m, s),$$

where $\mathfrak{p}(m, s)$ denotes the subspace spanned by the elements of the Hall basis of length $s$. Notice that

- $C^r(\mathfrak{n}_{m,k}) = \bigoplus_{s=r+1}^{k} \mathfrak{p}(m, s)$,
- $\mathfrak{p}(m, k) = \mathfrak{z}(\mathfrak{n}_{m,k})$.

The first assertion follows from the fact that every bracket of $r+1$ elements of $\mathfrak{n}_{m,k}$ is a linear combination of brackets of $r+1$ elements in the Hall basis (see proof of Theorem 3.1 in [11]). This implies $C^r(\mathfrak{n}_{m,k}) \subseteq \bigoplus_{s=r+1}^{k} \mathfrak{p}(m, s)$; the other inclusion is obvious. In particular, $\mathfrak{p}(m, k) = C^{k-1}(\mathfrak{n}_{m,k}) \subseteq \mathfrak{z}(\mathfrak{n}_{m,k})$. Now let $x \in \mathfrak{z}(\mathfrak{n}_{m,k})$ and let $e$ be a generator and assume $x \notin C^{k-1}(\mathfrak{n}_{m,k})$. Recall that $\mathfrak{n}_{m,k}$ is homomorphic image of the free Lie algebra $\mathfrak{f}_m$ so that there exist $X, E \in \mathfrak{f}_m$ such that $X \to x$ and $E \to e$, being $E$ a generator of $\mathfrak{f}_m$. Since $[x, e] = 0$ then $[X, E] = 0$ which says $X$ and $E$ are proportional, which is impossible (see for instance Ch. 2 in [1]). Thus $\mathfrak{p}(m, k) = \mathfrak{z}(\mathfrak{n}_{m,k})$.

Denote as $d_m(s)$ the dimension of $\mathfrak{p}(m, s)$. Inductively one gets [24]

$$s \cdot d_m(s) = m^s - \sum_{r|s, r<s} r \cdot d_m(r), \quad s \geqslant 1. \tag{2}$$

Hence for a fixed $m$, one has $d_m(1) = m$ and $d_m(2) = m(m-1)/2$.

**Example 2.2.** Given an ordered set of generators $e_1, \dots, e_m$ of a free 2-step nilpotent Lie algebra $\mathfrak{n}_{m,2}$, a Hall basis is

$$\mathcal{B} = \{e_i, [e_j, e_k]: i = 1, \dots, m, \ 1 \leqslant k < j \leqslant m\}. \tag{3}$$

Eq. (2) asserts that $\dim \mathfrak{n}_{m,2} = d_m(1) + d_m(2) = m + m(m-1)/2$. Since $\mathfrak{z}(\mathfrak{n}_{m,2}) = \mathfrak{p}(m,2)$, we have $\dim \mathfrak{z}(\mathfrak{n}_{m,2}) = m(m-1)/2$.

**Example 2.3.** For the free 3-step nilpotent Lie algebra on $m$ generators $\mathfrak{n}_{m,3}$ a Hall basis of a set of generators as before has the form

$$\mathcal{B} = \{e_i, [e_j, e_k], [[e_r, e_s], e_t]: i = 1, \dots, m, \ 1 \leqslant k < j \leqslant m, \ 1 \leqslant s < r \leqslant m, \ t \geqslant s\}. \tag{4}$$

It holds $\mathfrak{z}(\mathfrak{n}_{m,3}) = \mathfrak{p}(m,3)$, and so

$$\dim \mathfrak{z}(\mathfrak{n}_{m,3}) = d_m(3) = m(m^2 - 1)/3.$$

## 3. Free and free metabelian nilpotent Lie algebras and ad-invariant metrics

In this section we determine free nilpotent and free metabelian nilpotent Lie algebras admitting ad-invariant metrics.

Let $\mathfrak{g}$ denote a Lie algebra equipped with an ad-invariant metric $\langle\,,\rangle$, see (1). If $\mathfrak{m} \subseteq \mathfrak{g}$ is a subset, then we denote by $\mathfrak{m}^\perp$ the linear subspace of $\mathfrak{g}$ given by

$$\mathfrak{m}^\perp = \{x \in \mathfrak{g}, \ \langle x, v\rangle = 0 \text{ for all } v \in \mathfrak{m}\}.$$

In particular $\mathfrak{m}$ is called

- *isotropic* if $\mathfrak{m} \subseteq \mathfrak{m}^\perp$,
- *totally isotropic* if $\mathfrak{m} = \mathfrak{m}^\perp$, and
- *nondegenerate* if and only if $\mathfrak{m} \cap \mathfrak{m}^\perp = \{0\}$.

The proof of the next result follows easily from an inductive procedure.

**Lemma 3.1.** *Let $(\mathfrak{g}, \langle\,,\rangle)$ denote a Lie algebra equipped with an ad-invariant metric.*

(1) *If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ then $\mathfrak{h}^\perp$ is also an ideal in $\mathfrak{g}$.*
(2) *$C^r(\mathfrak{g})^\perp = C_r(\mathfrak{g})$ for all $r$.*

Thus on any Lie algebra admitting an ad-invariant metric the next equality holds

$$\dim \mathfrak{g} = \dim C^r(\mathfrak{g}) + \dim C_r(\mathfrak{g}). \tag{5}$$

For the case $r = 1$ one obtains

$$\dim \mathfrak{g} = \dim \mathfrak{z}(\mathfrak{g}) + \dim C^1(\mathfrak{g}). \tag{6}$$

**Example 3.2.** Let $\mathfrak{n}$ denote a 2-step nilpotent Lie algebra equipped with an ad-invariant metric. Assume $\mathfrak{z}(\mathfrak{n}) = C^1(\mathfrak{n})$, then by (6) the metric is neutral and $\dim \mathfrak{n} = 2 \dim \mathfrak{z}(\mathfrak{n})$. As a consequence the Heisenberg Lie algebra $\mathfrak{h}_n$ cannot be equipped with any ad-invariant metric.

Examples of nilpotent Lie algebras satisfying the equality (5) above for every $r$ arise by considering the semidirect product of a nilpotent Lie algebra $\mathfrak{n}$ with its dual space via de coadjoint representation $\mathfrak{n} \ltimes \mathfrak{n}^*$. The natural neutral metric on $\mathfrak{n} \ltimes \mathfrak{n}^*$ is ad-invariant.

Nevertheless, condition (6) (and hence (5)) is not sufficient for a 2-step nilpotent Lie algebra to admit an ad-invariant metric as shown for instance in [23].

**Lemma 3.3.** *Let $\mathfrak{g}$ denote a 2-step solvable Lie algebra provided with an ad-invariant metric, then $\mathfrak{g}$ is nilpotent and at most 3-step.*

**Proof.** Let $\langle\,,\rangle$ denote an ad-invariant metric on $\mathfrak{g}$. Since $\mathfrak{g}$ is 2-step solvable for all $x, y \in \mathfrak{g}'$ one has $[x, y] = 0$, which is equivalent to

$$\begin{aligned} 0 &= \big\langle [x, y], u \big\rangle \quad \text{for all } u \in \mathfrak{g}, \\ &= \big\langle [u, x], y \big\rangle \quad \text{for all } y \in \mathfrak{g}' \end{aligned}$$

thus $[u, x] \in [\mathfrak{g}, \mathfrak{g}]^{\perp} = \mathfrak{z}(\mathfrak{g})$, and since $x \in \mathfrak{g}'$, one has $[u, x] \in C^2(\mathfrak{g}) \subseteq \mathfrak{z}(\mathfrak{g})$ for all $u \in \mathfrak{g}$, $x \in \mathfrak{g}'$, that is $C^3(\mathfrak{g}) = 0$ and so $\mathfrak{g}$ is at most 3-step nilpotent. □

**Corollary 3.4.** *Let $\tilde{\mathfrak{n}}_{m,k}$ denote a free metabelian nilpotent Lie algebra admitting an ad-invariant metric, then $k \leqslant 3$.*

Remark 1 and the previous result say that a free metabelian nilpotent Lie algebra with an ad-invariant metric if it exists, is free nilpotent. Below we determine which free nilpotent Lie algebra admits such a metric.

Whenever $\mathfrak{n}_{m,k}$ is free nilpotent we have that $\dim \mathfrak{n}_{m,k}/C^1(\mathfrak{n}_{m,k}) = m$ so that

$$\dim \mathfrak{n}_{m,k} = m + \dim C^1(\mathfrak{n}). \tag{7}$$

Hence Eqs. (6) and (7) show that if $\mathfrak{n}_{m,k}$ admits an ad-invariant metric then

$$\dim \mathfrak{z}(\mathfrak{n}_{m,k}) = m. \tag{8}$$

**Proposition 3.5.** *If $\mathfrak{n}_{m,2}$ is a free 2-step nilpotent Lie algebra endowed with an ad-invariant metric, then $m = 3$.*

**Proof.** Let $\mathfrak{n}_{m,2}$ be the free 2-step nilpotent Lie algebra on $m$ generators. As we showed in Example 2.2 its center has dimension $m(m-1)/2$. Now if Eq. (8) holds then $m = 3$. □

**Proposition 3.6.** *Let $\mathfrak{n}_{m,3}$ be a free 3-step nilpotent Lie algebra provided with an ad-invariant metric, then $m = 2$.*

**Proof.** As shown in Example 2.3 the center $\mathfrak{z}(\mathfrak{n}_{m,3})$ has dimension $d_m(3) = m(m^2 - 1)/3$. From straightforward calculations, if Eq. (8) is satisfied then $m = 2$. □

**Proposition 3.7.** *No free $k$-step nilpotent Lie algebra $\mathfrak{n}_{m,k}$ on $m$ generators with $k \geqslant 4$ can be endowed with an ad-invariant metric.*

**Proof.** • 4-*Step nilpotent case*: In this case $\mathfrak{p}(m,4)=\mathfrak{z}(\mathfrak{n}_{m,4})$, thus from (2):

$$\dim \mathfrak{z}(\mathfrak{n}_{m,4}) = d_m(4) = \frac{1}{4}\big(m^4 - d_m(1) - 2d_m(2)\big) = \frac{m^2(m^2-1)}{4}.$$

Notice that for $m \geqslant 2$, one has $m^2(m^2-1)/4 > m$.

• *General case, $k \geqslant 5$*: Let $\mathfrak{n}_{m,k}$ denote the free $k$-step nilpotent Lie algebra on $m$ generators. The goal here is to show that for every $m$ and $k \geqslant 5$ the dimension of the center of $\mathfrak{n}_{m,k}$ is greater than $m$. In order to give a lower bound for $\dim \mathfrak{z}(\mathfrak{n}_{m,k})$ we construct elements of length $k$ in a Hall basis $\mathcal{B}$.

Let $\{e_1,\dots,e_m\}$ be a set of generators of $\mathfrak{n}_{m,k}$ and consider the set

$$\mathcal{U} = \big\{\big[\big[[e_i,e_j],e_k\big],e_m\big]:\ 1 \leqslant j < i \leqslant m,\ k \geqslant j\big\}.$$

Any element in $\mathcal{U}$ is basic and of length 4. Given $x \in \mathcal{U}$, the bracket

$$[x,e_m]^{(s)} := \big[\big[[x,\overbrace{e_m],e_m\big]\cdots,e_m}^{s}\big], \quad s \geqslant 1$$

is an element in the Hall basis if $\ell([x,e_m]^{(s)}) \leqslant k$.

In fact if $s=1$ then

(1) both $x=[[[e_i,e_j],e_k],e_m] \in \mathcal{U}$ and $e_m$ are elements of the Hall basis, and $x > e_m$ because of their length;
(2) also $x=[G,H]$ with $G=[[e_i,e_j],e_k]$ and $H=e_m$ and we have $e_m \geqslant H$.

So the conditions of the Hall basis definition are satisfied and hence $[x,e_m]^{(1)} \in \mathcal{B}$ and it belongs to $C^4(\mathfrak{n}_{m,k})$.

Inductively suppose $[x,e_m]^{(s-1)} \in \mathcal{B}$, then clearly $[[[[x,e_m],e_m]\cdots],e_m]^{(s-1)} > e_m$ and it is possible to write $[x,e_m]^{(s-1)} = [G,H]$ with $H=e_m$. Thus $[x,e_m]^{(s)} \in \mathcal{B}$. Notice that $[x,e_m]^{(s)} \in C^{s+3}(\mathfrak{n}_{m,k})$.

We construct the new set

$$\widetilde{\mathcal{U}} := \big\{[x,e_m]^{(k-4)}:\ x \in \mathcal{U}\big\} \subseteq C^{k-1}(\mathfrak{n}_{m,k}),$$

which is contained in the center of $\mathfrak{n}_{m,k}$ and it is linearly independent. Therefore

$$\dim \mathfrak{z}(\mathfrak{n}_{m,k}) \geqslant |\widetilde{\mathcal{U}}|. \tag{9}$$

Clearly $\widetilde{\mathcal{U}}$ and $\mathcal{U}$ have the same cardinality. Also, $|\mathcal{U}| = \sum_{j=1}^{m}(m-j+1)(m-j)$ since for every fixed $j=1,\dots,m$, the amount of possibilities to choose $k \geqslant j$ and $i > j$ is $(m-j+1)$ and $(m-j)$ respectively.

Straightforward computations give $|\widetilde{\mathcal{U}}| = 1/3m^3 + m^2 + 2/3m$ which combined with (9) proves that for any $m$ and $k \geqslant 5$

$$\dim \mathfrak{z}(\mathfrak{n}_{m,k}) \geqslant 1/3m^3 + m^2 + 2/3m.$$

The right hand side is greater than $m$ for all $m \geqslant 2$. According to (8), the free $k$-step nilpotent Lie algebra $\mathfrak{n}_{m,k}$ does not admit an ad-invariant metric if $k \geqslant 5$. $\square$

**Theorem 3.8.** *Let $\mathfrak{n}_{m,k}$ be the free $k$-step nilpotent Lie algebra on $m$ generators. Then $\mathfrak{n}_{m,k}$ admits an ad-invariant metric if and only if $(m,k)=(3,2)$ or $(m,k)=(2,3)$.*

**Proof.** Propositions 3.5, 3.6 and 3.7 prove that if $\mathfrak{n}_{m,k}$ admits an ad-invariant metric then $(m,k)=(3,2)$ or $(m,k)=(2,3)$. Let us show the resting part of the proof.

The Lie algebra $\mathfrak{n}_{3,2}$ has a basis $\{e_1,e_2,e_3,e_4,e_5,e_6\}$ with non-zero brackets

$$[e_1,e_2]=e_4, \qquad [e_1,e_3]=e_5, \qquad [e_2,e_3]=e_6. \tag{10}$$

Since $C^1(\mathfrak{n}_{3,2})=\mathfrak{z}(\mathfrak{n}_{3,2})$, if the metric $\langle\,,\rangle$ is ad-invariant the center of $\mathfrak{n}_{3,2}$ is totally isotropic, so it must hold

$$\langle e_i,e_j\rangle=0 \quad \text{for all } i,j=4,5,6.$$

The ad-invariance property says

$$\langle e_4,e_1\rangle=\big\langle[e_1,e_2],e_1\big\rangle=0$$

and similarly $\langle e_4,e_2\rangle=0$, therefore $\langle e_4,e_3\rangle\neq 0$. Analogously, $\langle e_5,e_2\rangle\neq 0$ and $\langle e_6,e_1\rangle\neq 0$. Moreover

$$\alpha=\big\langle[e_1,e_2],e_3\big\rangle=-\langle e_2,e_5,\rangle=\langle e_1,e_6\rangle.$$

Thus in the ordered basis $\{e_1,e_2,e_3,e_4,e_5,e_6\}$, a matrix of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & \alpha \\ a_{12} & a_{22} & a_{23} & 0 & -\alpha & 0 \\ a_{13} & a_{23} & a_{33} & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{with } a_{ij}\in K,\ \forall i,j=1,2,3 \text{ and } \alpha\neq 0 \tag{11}$$

corresponds to an ad-invariant metric on $\mathfrak{n}_{3,2}$.

The Lie algebra $\mathfrak{n}_{2,3}$ has a basis $\{e_1,e_2,e_3,e_4,e_5\}$ with non-zero Lie brackets

$$[e_1,e_2]=e_3, \qquad [e_1,e_3]=e_4, \qquad [e_2,e_3]=e_5. \tag{12}$$

Let $\langle\,,\rangle$ be an ad-invariant metric on $\mathfrak{n}_{2,3}$. Then $\mathfrak{z}(\mathfrak{n}_{3,2})=C^2(\mathfrak{n}_{2,3})=span\{e_4,e_5\}$ while $C^1(\mathfrak{n}_{2,3})=C^2(\mathfrak{n}_{2,3})\oplus Ke_3$.

The ad-invariance property also says that

$$0=\langle e_4,e_3\rangle=\langle e_4,e_4\rangle=\langle e_4,e_5\rangle=\langle e_5,e_3\rangle=\langle e_5,e_5\rangle.$$

Moreover

$$\langle e_1,e_3\rangle=\big\langle e_1,[e_1,e_2]\big\rangle=0 \quad \text{and} \quad \langle e_2,e_3\rangle=\big\langle e_2,[e_1,e_2]\big\rangle=0;$$
$$\langle e_1,e_4\rangle=\big\langle e_1,[e_1,e_3]\big\rangle=0 \quad \text{and} \quad \langle e_2,e_5\rangle=\big\langle e_2,[e_2,e_3]\big\rangle=0.$$

Therefore

$$\langle e_3,e_3\rangle=\big\langle[e_1,e_2],e_3\big\rangle=-\langle e_2,e_4\rangle=\langle e_1,e_5\rangle=\alpha\neq 0,$$

which amounts to the following matrix for $\langle\,,\rangle$ in the ordered basis above:

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 & \alpha \\ a_{12} & a_{22} & 0 & -\alpha & 0 \\ 0 & 0 & \alpha & 0 & 0 \\ 0 & -\alpha & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{with } a_{ij} \in K,\ \forall i, j = 1, 2 \text{ and } \alpha \neq 0. \qquad \square \tag{13}$$

**Remark 2.** The free nilpotent Lie algebras above can be constructed as extensions of abelian Lie algebras. This is the way in which they appear in [9], where Favre and Santharoubane obtained the classification of the nilpotent Lie algebras of dimension $\leqslant 7$ admitting an ad-invariant metric. According to their results, any of the Lie algebras equipped with an ad-invariant metric $\mathfrak{n}_{3,2}$ or $\mathfrak{n}_{2,3}$ as above is equivalent to one of the followings

$$(\mathfrak{n}_{3,2}, B_{3,2}): \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (\mathfrak{n}_{2,3}, B_{2,3}): \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{14}$$

Remark 1, Corollary 3.4 and the previous theorem imply the next result.

**Theorem 3.9.** *Let $\tilde{\mathfrak{n}}_{m,k}$ be the free metabelian $k$-step nilpotent Lie algebra on $m$ generators. Then $\tilde{\mathfrak{n}}_{m,k}$ admits an ad-invariant metric if and only if $(m,k) = (3,2)$ or $(m,k) = (2,3)$.*

**Remark 3.** Theorems 3.8 and 3.9 can be obtained in reverse order. One can start by performing the study of the dimension of the center of free metabelian nilpotent Lie algebras. The center of $\tilde{\mathfrak{n}}_{m,k}$ is also spanned by commutators of length $k$ and its dimension is well known (see, for instance, [1]). So for the free metabelian nilpotent case similar arguments to those in the proof of Theorem 3.8 give Theorem 3.9.

Now, since the quotient map $\mathfrak{n}_{m,k} \mapsto \tilde{\mathfrak{n}}_{m,k}$ maps $\mathfrak{z}(\mathfrak{n}_{m,k})$ onto $\mathfrak{z}(\tilde{\mathfrak{n}}_{m,k})$, the dimension of $\mathfrak{z}(\mathfrak{n}_{m,k})$ is greater or equal than the dimension of $\mathfrak{z}(\tilde{\mathfrak{n}}_{m,k})$. Hence Theorem 3.8 follows.

However this approach would miss the result in Lemma 3.3 which impose a strong condition.

## 4. The automorphisms groups of $\mathfrak{n}_{3,2}$ and $\mathfrak{n}_{2,3}$

Here we study the automorphisms of the Lie algebras in Theorem 3.8. This is indeed a topic of active research (see for instance [7,8] and references therein). Our goal is to write explicitly the algebraic structure in terms of the actions and representations of the different subgroups or subalgebras. We also distinguish the subgroup of orthogonal automorphisms (resp. the Lie algebra of skew-symmetric derivations) in presence of the ad-invariant metric fixed in (14).

Recall that a derivation of a Lie algebra $\mathfrak{g}$ is a linear map $t : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$t[x, y] = [tx, y] + [x, ty] \quad \text{for all } x, y \in \mathfrak{g}.$$

Whenever $\mathfrak{g}$ is endowed with a metric $\langle\,,\rangle$ a skew-symmetric derivation of $\mathfrak{g}$ is a derivation $t$ such that

$$\langle ta, b\rangle = -\langle a, tb\rangle \quad \text{for all } a, b \in \mathfrak{g}. \tag{15}$$

We denote by $\mathrm{Der}(\mathfrak{g})$ the Lie algebra of derivations of $\mathfrak{g}$, which is the Lie algebra of the group of automorphisms of $\mathfrak{g}$, $\mathrm{Aut}(\mathfrak{g})$. Let $\mathrm{Dera}(\mathfrak{g})$ denote the subalgebra of $\mathrm{Der}(\mathfrak{g})$ consisting of skew-symmetric derivations of $(\mathfrak{g}, \langle\, ,\,\rangle)$. Thus $\mathrm{Dera}(\mathfrak{g})$ is the Lie algebra of the group of orthogonal automorphisms denoted by $\mathrm{Auto}(\mathfrak{g})$:

$$\mathrm{Auto}(\mathfrak{g}) = \big\{\alpha \in \mathrm{Aut}(\mathfrak{g}):\ \langle \alpha x, \alpha y\rangle = \langle x, y\rangle,\ \forall x, y \in \mathfrak{g}\big\}.$$

For an arbitrary Lie algebra $\mathfrak{g}$, each $t \in \mathrm{Aut}(\mathfrak{g})$ leaves invariant both the commutator ideal $C^1(\mathfrak{g})$ and the center $\mathfrak{z}(\mathfrak{g})$. Thus $t$ induces automorphisms of the quotient Lie algebras $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ and $\mathfrak{g}/C^1(\mathfrak{g})$.

In particular if $\mathfrak{g}$ is solvable, $\mathfrak{g}/C^1(\mathfrak{g})$ is a nontrivial abelian Lie algebra. Hence any $t \in \mathrm{Aut}(\mathfrak{g})$ induces an element $s \in \mathrm{Aut}(\mathfrak{g}/C^1(\mathfrak{g}))$ which as an automorphism of an abelian Lie algebra, $s \in \mathrm{GL}(p, K)$ where $p = \dim(\mathfrak{g}/C^1(\mathfrak{g}))$. Note that if $\mathfrak{g}$ is free nilpotent, $p$ coincides with the number of generators of $\mathfrak{g}$.

Below we proceed to the study in each case. More computations can be found in [6] and for the general cases we refer to [7,8].

### 4.1. *The Lie algebra* $\mathfrak{n}_{2,3}$

Let $t$ denote an automorphism of $\mathfrak{n}_{2,3}$ and let $e_1, e_2, e_3, e_4, e_5$ be the basis given in (12). Since $t$ leaves invariant the commutator and the center and $C^1(\mathfrak{n}_{2,3}) = span\{e_3, e_4, e_5\}$ and $\mathfrak{z}(\mathfrak{n}_{2,3}) = span\{e_4, e_5\}$, we have that $t_{ij} = 0$ if $i = 1, 2,\ j = 3, 4, 5$.

Notice that the Lie algebra $\mathfrak{n}_{2,3}/\mathfrak{z}(\mathfrak{n}_{2,3})$ is isomorphic to the Heisenberg Lie algebra $\mathfrak{h}_1$, hence $t$ induces an automorphism $\bar{t} \in \mathrm{Aut}(\mathfrak{h}_1)$.

Consider the following matrices in $\mathrm{Aut}(\mathfrak{n}_{2,3})$:

$$\mathcal{G} = \left\{ \tilde{A} = \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{$A$}} & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ 0 & 0 & \det(A) & 0 & 0 \\ 0 & 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\det(A)A$}} \\ 0 & 0 & 0 & & \end{pmatrix},\ A \in \mathrm{GL}(2, K) \right\},$$

$$\mathcal{H} = \left\{ h_{(x,y,z)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ y & x & 1 & 0 & 0 \\ z + \frac{1}{2}xy & \frac{1}{2}x^2 & x & 1 & 0 \\ -\frac{1}{2}y^2 & z - \frac{1}{2}xy & -y & 0 & 1 \end{pmatrix},\ (x, y, z) \in K^3 \right\},$$

$$\mathcal{R} = \left\{ r_{(u,v,w)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ v & w & 0 & 1 & 0 \\ u & -v & 0 & 0 & 1 \end{pmatrix},\ (u, v, w) \in K^3 \right\}.$$

Note that $\mathcal{G}$, $\mathcal{R}$ and $\mathcal{H}$ are subgroups of $\mathrm{Aut}(\mathfrak{n}_{2,3})$.

Moreover every $t \in \mathrm{Aut}(\mathfrak{n}_{2,3})$ can be written as a product of matrices

$$t = \tilde{A} \cdot r_{(u,v,w)} \cdot h_{(x,y,z)}, \quad \text{with } \tilde{A} \in \mathcal{G},\ r_{(u,v,w)} \in \mathcal{R},\ h_{(x,y,z)} \in \mathcal{H}.$$

The elements of $\mathcal{H}$ commute with those of $\mathcal{R}$; also $\mathcal{H} \cap \mathcal{R} = \{I\}$. Hence $\mathcal{R} \cdot \mathcal{H} \simeq \mathcal{R} \times \mathcal{H}$.

It holds $h_{(x,y,z)} \cdot h_{(x',y',z')} = h_{(x+x', y+y', z+z'+\frac{1}{2}(xy'-x'y))}$, where $\cdot$ denotes the product of matrices. Thus the map from the Heisenberg Lie group $H_1$ to $\mathrm{Aut}(\mathfrak{n}_{2,3})$ given by $(x, y, z) \mapsto h_{(x,y,z)}$ is an isomorphism of groups.

Analogously $(u, v, w) \mapsto r_{(u,v,w)}$ is an isomorphism of groups from $K^3$ to $\mathcal{R} \subseteq \mathrm{Aut}(\mathfrak{n}_{2,3})$.

The action by conjugation of $\mathcal{G}$ preserves both $\mathcal{H}$ and $\mathcal{R}$. Thus the map

$$\tau_{\tilde{A}}(r, h) = \left(\tilde{A} r \tilde{A}^{-1}, \tilde{A} h \tilde{A}^{-1}\right),$$

defines a group homomorphism from $\mathcal{G}$ to $\mathrm{Aut}(\mathcal{R} \times \mathcal{H})$.

The subgroup $\mathcal{R} \times \mathcal{H}$ is normal in $\mathrm{Aut}(\mathfrak{n}_{2,3})$ and $\mathcal{G} \cap (\mathcal{R} \times \mathcal{H}) = \{I\}$, hence $\mathrm{Aut}(\mathfrak{n}_{2,3}) \simeq \mathcal{G} \ltimes_\tau (\mathcal{R} \times \mathcal{H})$ [15]. It is clear that $\mathcal{G} \simeq \mathrm{GL}(2, K)$, $\mathcal{R} \simeq K^3$ and $\mathcal{H} \simeq H_1$ and these isomorphisms preserve the action of $\mathrm{GL}(2, K)$ in $H_1$ and $K^3$. So the next result follows and the additional orthogonal condition leads the second proof.

**Proposition 4.1.** *The group of automorphisms of* $\mathfrak{n}_{2,3}$ *is*

$$\mathrm{Aut}(\mathfrak{n}_{2,3}) \simeq \mathrm{GL}(2, K) \ltimes \left(K^3 \times H_1\right),$$

*where* $H_1$ *denotes the Heisenberg Lie group of dimension three.*

*The group of orthogonal automorphisms of* $(\mathfrak{n}_{2,3}, B_{2,3})$ *is*

$$\mathrm{Auto}(\mathfrak{n}_{2,3}) \simeq \mathcal{S} \ltimes H_1,$$

*where* $\mathcal{S}$ *is the subgroup of* $\mathcal{G}$ *consisting of the matrices* $\tilde{A}$ *with* $A \in \mathrm{GL}(2, K)$, $\det(A) = \pm 1$, *where the action is the restriction of the action of* $\mathcal{G}$ *in the Heisenberg Lie group* $H_1$ *described before.*

The next result derives from the previous one by looking at the Lie algebras.

**Corollary 4.2.** *The Lie algebra of derivations of* $\mathfrak{n}_{2,3}$ *is isomorphic to*

$$\mathfrak{gl}(2, K) \ltimes \left(\mathfrak{h}_1 \times K^3\right).$$

*In particular, fix the ad-invariant metric* $\langle\,,\rangle$ *in* $(\mathfrak{n}_{2,3}, B_{2,3})$ *given by the matricial representation as in* (14)*.*

*The set of skew-symmetric derivations is represented by the Lie algebra*

$$\mathrm{Dera}(\mathfrak{n}_{2,3}) \simeq \mathfrak{sl}(2, K) \ltimes \mathfrak{h}_1$$

*while the set of inner derivations is* (*isomorphic to*) $\mathfrak{h}_1$.

A brief description of the actions in the semidirect products in the previous corollary is as follows.

The subalgebra $\mathfrak{h}_1 \times K^3$ is an ideal of the Lie algebra of derivations namely the radical. The action of $\mathfrak{gl}(2, K)$ on this ideal preserves each $\mathfrak{h}_1$ and $K^3$ respectively.

The subalgebra $\mathfrak{sl}(2, K)$ of $\mathfrak{gl}(2, K)$ acts on $\mathfrak{h}_1$ by derivations. Recall that $\mathfrak{sl}(2, K) \simeq \mathrm{Der}(\mathfrak{h}_1)$.

Moreover, the action of $\mathfrak{sl}(2, K)$ on $K^3$ is given by its irreducible representation in dimension three (see [12]).

For a suitable basis $\{T, E, F, H\}$ of $\mathfrak{gl}(2, K)$ such that *span*$\{E, F, H\} \simeq \mathfrak{sl}(2, K)$ it holds that the action of $T$ on $\mathfrak{h}_1$ is diagonal and on $K^3$ is twice the identity.

### 4.2. *The Lie algebra $\mathfrak{n}_{3,2}$*

With the usual conventions, denote by $E_{ij}$ the $6 \times 6$ matrix which has a 1 in the file $i$ and column $j$ and 0 otherwise. Let $T$ and $f_i$, $i = 1, \ldots, 8$, be the following matrices

$$T = E_{11} + E_{22} + E_{33} + 2E_{44} + 2E_{55} + 2E_{66},$$

$$\begin{aligned} f_1 &= E_{11} - E_{33} + E_{44} - E_{66}, & f_5 &= E_{13} - E_{46}, \\ f_2 &= E_{22} - E_{33} + E_{44} - E_{55}, & f_6 &= E_{31} - E_{64}, \\ f_3 &= E_{12} + E_{56}, & f_7 &= E_{23} + E_{45}, \\ f_4 &= E_{21} + E_{65}, & f_8 &= E_{32} + E_{54}. \end{aligned}$$

By considering these matrices as endomorphisms in the basis $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ of $\mathfrak{n}_{3,2}$ given in (10) it is easy to see that they are derivations of the Lie algebra.

With the Lie bracket of matrices, the vector space spanned by $f_1, \ldots, f_8$ constitute a Lie algebra isomorphic to $\mathfrak{sl}(3, K)$, such that $[T, f_i] = 0$ for all $i = 1, \ldots, 8$.

For $i = 1, \ldots, 9$, let $A_i$ denote the matrices

$$\begin{aligned} A_1 &= E_{52}, & A_2 &= E_{63}, & A_3 &= -E_{41} - E_{63}, \\ A_4 &= E_{61}, & A_5 &= E_{51} + E_{62}, & A_6 &= E_{51} - E_{62}, \\ A_7 &= -E_{42} + E_{53}, & A_8 &= E_{42} + E_{53}, & A_9 &= E_{43}, \end{aligned} \tag{16}$$

which are also derivations of $\mathfrak{n}_{2,3}$. By canonical computations one determines that the matrices $T$, $f_i$, $A_j$ generate the Lie algebra $\mathrm{Dera}(\mathfrak{n}_{3,2})$.

The matrices in (16) generate an abelian Lie algebra $\mathfrak{N}$ of dimension nine. Actually this is a faithful representation of minimal dimension of the Lie algebra $K^9$ (see for instance [3]).

The action of $\mathfrak{sl}(3, K)$ on $K^9$ is given by the adjoint representation $f_i \cdot A_j = [f_i, A_j]$ for all $i$, $j$, while the action of $T$ on $K^9$ is represented by the identity map.

**Proposition 4.3.** *Let $\mathfrak{n}_{3,2}$ denote the free 2-step nilpotent Lie algebra on three generators. The Lie algebra of derivations of $\mathfrak{n}_{3,2}$ is*

$$\mathrm{Der}(\mathfrak{n}_{3,2}) = \mathfrak{gl}(3, K) \ltimes K^9.$$

*The set of skew-symmetric derivations of $(\mathfrak{n}_{3,2}, B_{3,2})$ is given by*

$$\mathrm{Dera}(\mathfrak{n}_{3,2}) \simeq \mathfrak{sl}(3, K) \ltimes K^3,$$

*while the set of inner derivations is isomorphic to $K^3$.*


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